Table 1 ~~(cont.)~~ FOEC

R	N	RII	RI	HII	HI	TII	CII	I
	Triclinic	Trigonal		Hexagonal		Tetragonal	Cubic	Isotropic
	$1(C_1)$ $\bar{1}(S_2)$	$3(C_3)$, $\bar{3}(C_{3i})$	$3m(C_{3v})$, $32(D_3)$, $\bar{3}\frac{2}{m}(D_{3d})$	$6(C_6)$, $\bar{6}(C_{3h})$, $\frac{6}{m}(C_{6h})$	$6m2(D_{3h})$, $6mm(C_{6v})$, $622(D_6)$, $\frac{622}{mmm}(D_{6h})$	$4(C_4)$ $\bar{4}(S_4)$ $\frac{4}{m}(C_{4h})$	$23(T)$ $\frac{2}{m}\bar{3}(T_h)$	
(1)	126 (2)	42 (3)	28 (4)	24 (5)	19 (6)	36 (7)	14 (8)	4 (9)
1	1111	---	---	---	---	--	--	---
4	1112	---	---	---	---	--	--	---
4	1113	---	---	---	---	--	--	1112
8	1114	---	---	0	0	0	0	0
8	1115	---	0	0	0	0	0	0
8	1116	---	0	---	0	--	0	0
6	1122	---	---	---	---	--	--	---
12	1123	---	---	---	---	--	--	---
24	1124	---	---	0	0	0	0	0
24	1125	---	0	0	0	0	0	0
24	1126	-1116	0	-1116	0	--	0	0
6	1133	---	---	---	---	--	1122	1122
24	1134	---	---	0	0	0	0	0
24	1135	---	0	0	0	0	0	0
24	1136	---	0	---	0	--	0	0
24	1144	---	---	---	---	--	--	2.1122-1123
48	1145	---	0	---	0	--	0	0
48	1146	---	0	0	0	0	0	0
24	1155	---	---	---	---	--	--	4.1111-1123
48	1156	---	---	0	0	0	0	0
24	1166	---	---	---	---	--	--	4.1111-1112
4	1222	(8.1111+7.1112 -2.1166)/9	(8.1111+7.1112 -2.1166)/9	(8.1111+7.1112 -2.1166)/9	(8.1111+7.1112 -2.1166)/9	1112	1112	1112
12	1223	3.1113+1123 -3.2223	3.1113+1123 -3.2223	3.1113+1123 -3.2223	3.1113+1123 -3.2223	1123	1123	1123
24	1224	-(2.1114+3.1124 -1156)/3	-(2.1114+3.1124 -1156)/3	0	0	0	0	0
24	1225	-(2.1115+3.1125 +1146)/3	0	0	0	0	0	0
24	1226	-1116	0	-1116	0	-1126	0	0
12	1233	---	---	---	---	--	1123	1123
48	1234	---	---	0	0	0	0	0
48	1235	---	0	0	0	0	0	0
48	1236	-2.1136	0	-2.1136	0	0	0	0
48	1244	---	---	---	---	--	--	3.1112-1123
96	1245	---	0	---	0	0	0	0
96	1246	-2(1115+1125)	0	0	0	0	0	0
48	1255	---	---	---	---	1244	--	3.1112-1123
96	1256	2(1114+1124)	2(1114+1124)	0	0	0	0	0
48	1266	4.1111+2.1112 -2.1122	4.1111+2.1112 -2.1122	4.1111+2.1112 -2.1122	4.1111+2.1112 -2.1122	--	--	4.1111+2.1112 -2.1122
4	1333	---	---	---	---	--	1112	1112
24	1334	---	---	0	0	0	0	0
24	1335	---	0	0	0	0	0	0
24	1336	0	0	0	0	--	0	0
48	1344	---	---	---	---	--	1255	3.1112-1123
96	1345	---	0	---	0	--	0	0
96	1346	-2.1135-3.1235	0	0	0	0	0	0
48	1355	---	---	---	---	--	1266	1126
96	1356	2.1134	2.1134	0	0	0	0	0
48	1366	-6.1113-1123 +9.2223	6.1113-1123 +9.2223	6.1113-1123 +9.2223	6.1113-1123 +9.2223	--	1244	3.1112-1123
32	1444	---	---	0	0	0	0	0
96	1445	---	0	0	0	0	0	0
96	1446	(2.1145+3.1245)/2	0	(2.1145+3.1245)/2	0	--	0	0
96	1455	---	---	0	0	0	0	0
192	1456	-2.1144+2.1155 -3.1244+3.1255	-2.1144+2.1155 -3.1244+3.1255	-2.1144+2.1155 -3.1244+3.1255	-2.1144+2.1155 -3.1244+3.1255	--	--	-2.1144+2.1155 -3.1244+3.1255
96	1466	-1114-1124 +1156	-1114-1124 +1156	0	0	0	0	0
32	1555	---	0	0	0	0	0	0
96	1556	-(2.1145+3.1245)/2	0	-(2.1145+3.1245)/2	0	--	0	0
96	1566	-1115-1125 -1146	0	0	0	0	0	0
32	1666	-4.1116/3	0	0	0	--	0	0
1	2222	(5.1111+1112 +1166)/9	0	(5.1111+1112 +1166)/9	0	1111	1111	1111
4	2223	---	---	---	---	1113	1112	1112
8	2224	-(1114+1156)/3	-(1114+1156)/3	0	0	0	0	0
8	2225	-(1115-1146)/3	0	0	0	0	0	0
8	2226	1116	0	1116	0	-1116	0	0
6	2233	1133	1133	1133	1133	1133	1122	1122
24	2234	-1134-1234	-1134-1234	0	0	0	0	0
24	2235	-1135-1235	0	0	0	0	0	0
20	2236	1136	0	1136	0	-1136	0	0
24	2244	(2.1155-1244 +1255)/2	(2.1155-1244 +1255)/2	(2.1155-1244 +1255)/2	(2.1155-1244 +1255)/2	1155	1166	4.1111-1112
48	2245	-1145-1245	0	-1145-1245	0	-1145	0	0
48	2246	-(8.1115+1146)/3	0	0	0	0	0	0
24	2255	(2.1144+1244 -1255)/2	(2.1144+1244 -1255)/2	(2.1144+1244 -1255)/2	(2.1144+1244 -1255)/2	1144	1144	2.1122-1123
48	2256	(8.1114-1156)/3	(8.1114-1156)/3	0	0	0	0	0
24	2266	(16.1111-4.1112 -1166)/3	(16.1111-4.1112 -1166)/3	(16.1111-4.1112 -1166)/3	(16.1111-4.1112 -1166)/3	1166	1155	4.1111
4	2333	1333	1333	1333	1333	1333	1113	1113
24	2334	-1334	-1334	0	0	0	0	0
24	2335	-1335	0	0	0	0	0	0
24	2336	0	0	0	0	-1336	0	0
48	2344	1355	1355	1355	1355	1355	1266	4.1111+2.1112 -2.1122
96	2345	-1345	0	-1345	0	-1345	0	0

The invariance property of the strain energy thus leads to a system of equations with the values of m and n given by:

$$\text{Trigonal: } m=-\tfrac{1}{2},\ n=\frac{\sqrt{3}}{2} \quad \text{or} \quad m=\tfrac{1}{2},\ n=-\frac{\sqrt{3}}{2}$$

$$\text{Hexagonal: } m=\tfrac{1}{2},\ n=\frac{\sqrt{3}}{2} \quad \text{or} \quad m=-\tfrac{1}{2},\ n=-\frac{\sqrt{3}}{2}\,.$$

Because of the invariance of η_3 in equations (3), all the FOEC with an added index '3' satisfy the equations (A1)–(A10) for TOEC given by Hearmon (1953). Therefore, only five additional sets of equations are needed for the remaining FOEC not containing the index '3'. These equations (B1)–(B5) are given in the Appendix. For the hexagonal system, all the terms in equations (B2) and (B4) are set to zero owing to the symmetry operations.

The final results for FOEC calculated in this way for *R*I, *R*II, *H*I, and *H*II are given in Table 1. The column headings, reading from top to bottom, are (1) name of system (and its short form), (2) the Hermann–Mauguin and Schönflies symbols of the classes, (3) the number of independent constants and (4) column number. While the numerals in column (2) represent the FOEC in a triclinic system, they also serve as a list of all constants for other crystals whether they are independent or not. The independent ones in other crystal classes will be indicated by a bar and the dependent ones are expressed in terms of them. In this way, the independent number of FOEC for a Laue group is just the number of bars in that column. With the Table given by Ghate (1965) for Laue groups *N*, *M*, *O*, *C*I, and *T*I, the scheme for FOEC is now complete. The ratio R is defined as $R=C_{pqrs}/C_{ijklmnot}$ (for FOEC), where $pqrs$ and $ijklmnot$ are single and double indices respectively. The sum of all values of R for nth order elastic constants should be 3^{2n}. This can be used as a double check in preparing the Tables 1–3.

The FOEC for an isotropic system can be obtained by combining the cubic system (*C*II) with hexagonal system (*H*II). The result gives the four independent constants as 1111, 1112, 1122, and 1123. The equations relating the different constants for an isotropic system agree with those given by Krishnamurty (1963).

Calculations for FFOEC and SOEC

The direct-inspection method employed here is the same as that used for second-, third-, and fourth-order constants (Fumi, 1951, 1952*a*, *b*, *c*, 1953; Hearmon, 1953; Ghate, 1964, 1965). The results of FFOEC and SOEC are presented in Tables 2 and 3. These Tables are presented in the similar manner to Table 1.

In principle the FFOEC and SOEC could be worked out for the other Laue groups as we did for FOEC in the previous section except for the formidable algebra involved. A computer should be ultilized for this lengthy calculation.

To illustrate the use of this kind of table, we can write the terms of the elastic energy ϕ_5 for a cubic crystal. To do this, we first list the resulting 18 independent FFOEC and the equivalent coefficients in Table 4. From Table 4 the elastic energy ϕ_5 for a cubic crystal can then be written. The result is the sum of all the terms given in Table 5 divided by 5!. This expression has many more terms than of the lower order. The terms in ϕ_4 and ϕ_6 of other crystals can be worked out in the same way.

Applications

With the continuing improvement of the experimental accuracy in velocity measurement and the development of the method of shock waves (Graham, 1972), the determination of higher-order elastic constants becomes possible for all types of crystals. The contribution of higher-order terms in the experiments involv-

Table 2. *Fifth-order elastic constants* (*FFOEC*)

R	N Tri-clinic		M Mono-clinic		O Ortho-rhombic	TI Tetragonal	TII	CI Cubic	CII
	1 $\bar{1}$		2 2/m m		222 mmm	4mm $\bar{4}$2m 422 4/mmm	4 $\bar{4}$ 4/m	$\bar{4}$3m 432 m3m	23 $\frac{2}{m}\bar{3}$
		m=x₂x₃ C₂=x₁	m=x₃x₁ C₂=x₂	m=x₁x₂ C₂=x₃					
	252	136	136	136	78	44	68	18	26
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	11111	---	---	---	---	---	---	---	---
5	11112	---	---	---	---	---	---	---	---
5	11113	---	---	---	---	---	---	11112	---
10	11114	---	0	0	0	0	0	0	0
10	11115	0	---	0	0	0	0	0	0
10	11116	0	0	---	0	0	---	0	0
10	11122	---	---	---	---	---	---	---	---
20	11123	---	---	---	---	---	---	---	---
40	11124	---	0	0	0	0	0	0	0
40	11125	0	---	0	0	0	0	0	0
40	11126	0	0	---	0	0	---	0	0
10	11133	---	---	---	---	---	---	11122	---
40	11134	---	0	0	0	0	0	0	0
40	11135	0	---	0	0	0	0	0	0
40	11136	0	0	---	0	0	---	0	0
40	11144	---	---	---	---	---	---	---	---
80	11145	0	0	---	0	0	---	0	0
80	11146	0	---	0	0	0	0	0	0
40	11155	---	---	---	---	---	---	---	---
80	11156	---	0	0	0	0	0	0	0
40	11166	---	---	---	---	---	---	11155	---
10	11222	---	---	---	---	11122	11122	11122	11122